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Interaction of a normal load with a bonded circular punch

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Abstract. The problem of a flat circular punch bonded to a transversely isotropic elastic half-space and interacting with an arbitrarily located normal concentrated load is considered. The closed form exact solution is obtained for the linear and angular displacements of the punch. The solution is based on the results previously obtained by the author and combined with the reciprocal theorem. A numerical example is presented, in order to compare the linear and angular displacements of a smooth punch with the similar parameters for a bonded punch.

1. Introduction

The bonded punch problem belongs to the class of mixed-mixed problems of elasticity theory which are among the most complicated due to the coupling between the normal and tangential parameters. We should mention the works of Mossakovskii (1954) and Ufliand (1956) among the first published exact solutions for the case of an *isotropic* half-space, obtained by using various integral transforms. A more compact solution has been reported by Kapshivyi and Masliuk (1967), who used a special apparatus of p-analytical functions. The first *elementary* exact solution for a *transversely isotropic* elastic half-space was published by Fabrikant (1971a). Four different types of solution of the governing set of integral equations were reported by Fabrikant (1986). The problem of a bonded circular punch subjected to a shifting force and a tilting moment was first considered by Fabrikant (1971b).

The problem of interaction between a bonded punch and an internal point force is extremely important in engineering. First of all, the point force solution can be used as a Green function for solving more complicated problems of various distributed loadings. Secondly, the problem is of value in its own right. For example, in geotechnical engineering it can be viewed as the problem of interaction between a rigid foundation and an anchor load. The problem is very complicated. Only some particular cases were considered until now: the case of a surface load outside a bonded punch was considered by Fabrikant (1975), and an axisymmetric case of a loading under a punch was solved by Fabrikant and Sankar (1986). The general case of an arbitrarily located load is considered here for the first time. The solution has become possible due to the new results in potential theory obtained by the author (Fabrikant, 1989). Numerical results are obtained in order to compare the linear and angular displacements of a bonded punch due to a concentrated load with similar results for a smooth punch.

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2. Theory

Consider a transversely isotropic elastic body which is characterised by five elastic constants A_{ik} defining the following stress-strain relationships:

$$
\begin{aligned}
\sigma_x &= A_{11}\frac{\partial u_x}{\partial x} + (A_{11} - 2A_{66})\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z} \\
\sigma_y &= (A_{11} - 2A_{66})\frac{\partial u_x}{\partial x} + A_{11}\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z} \\
\sigma_z &= A_{13}\frac{\partial u_x}{\partial x} + A_{13}\frac{\partial u_y}{\partial y} + A_{33}\frac{\partial w}{\partial z} \\
\tau_{xy} &= A_{tt}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad \tau_{yz} = A_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}\right) \\
\tau_{zx} &= A_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u_x}{\partial z}\right).
\end{aligned} \tag{1}
$$

The equilibrium equations are

$$
\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \\
&\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0.
\end{aligned} \tag{2}
$$

Substitution of (1) in (2) yields

$$
\begin{aligned}
&A_{11}\frac{\partial^2 u_x}{\partial x^2} + A_{66}\frac{\partial^2 u_x}{\partial y^2} + A_{44}\frac{\partial^2 u_x}{\partial z^2} + (A_{11} - A_{66})\frac{\partial^2 u_y}{\partial x \partial y} + (A_{13} + A_{44})\frac{\partial^2 w}{\partial x \partial z} = 0 \\
&A_{66}\frac{\partial^2 u_y}{\partial x^2} + A_{11}\frac{\partial^2 u_y}{\partial y^2} + A_{44}\frac{\partial^2 u_y}{\partial z^2} + (A_{11} - A_{66})\frac{\partial^2 u_x}{\partial x \partial y} + (A_{13} + A_{44})\frac{\partial^2 w}{\partial y \partial z} = 0 \\
&A_{44}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + A_{33}\frac{\partial^2 w}{\partial z^2} + (A_{44} + A_{13})\left(\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z}\right) = 0.
\end{aligned} \tag{3}
$$

Introduce complex tangential displacements $u = u_x + \mathrm{i}u_y$, and $\bar{u} = u_x - \mathrm{i}u_y$. This will allow us to reduce the number of equations (3) by one, and to rewrite these equations in a more compact manner, namely

$$
\begin{aligned}
&\tfrac{1}{2}(A_{11} + A_{66})\Delta u + A_{44}\frac{\partial^2 u}{\partial z^2} + \tfrac{1}{2}(A_{11} - A_{66})\Lambda^2 \bar{u} + (A_{13} + A_{44})\Lambda\frac{\partial w}{\partial z} = 0 \\
&A_{44}\Delta w + A_{33}\frac{\partial^2 w}{\partial z^2} + \tfrac{1}{2}(A_{13} + A_{44})\frac{\partial}{\partial z}(\bar{\Lambda} u + \Lambda \bar{u}) = 0.
\end{aligned} \tag{4}
$$

Here the following differential operators were used:

$$
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \Lambda = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} \tag{5}
$$

and the overbar everywhere indicates the complex conjugate value. Note also that $\Delta=\Lambda\bar{\Lambda}$. One can verify that equations (4) can be satisfied by

$$u=\Lambda(F_1+F_2+\mathrm{i}F_3) \qquad w=m_1\frac{\partial F_1}{\partial z}+m_2\frac{\partial F_2}{\partial z} \tag{6}$$

where all three functions F_k satisfy the equation (Elliott 1948)

$$\Delta F_k+\gamma_k^2\frac{\partial^2 F_k}{\partial z^2}=0 \qquad \text{for } k=1,2,3 \tag{7}$$

and the values of m_k and γ_k are related by the following expressions (Elliott 1948):

$$\begin{gathered}\frac{A_{44}+m_k(A_{13}+A_{44})}{A_{11}}=\frac{m_kA_{33}}{m_kA_{44}+A_{13}+A_{44}}=\gamma_k^2 \qquad \text{for } k=1,2\\ \gamma_3=(A_{44}/A_{66})^{1/2}.\end{gathered} \tag{8}$$

Introducing the notation $z_k=z/\gamma_k$, for $k=1,2,3$, we may call function $F_k=F(x,y,z_k)$ harmonic. Note the property $m_1m_2=1$, which seems to have escaped the attention of previous researchers, and which will help us to simplify various expressions to follow. The other elastic constants which will be used throughout the paper are

$$\begin{gathered}G_1=\beta+\gamma_1\gamma_2H \qquad G_2=\beta-\gamma_1\gamma_2H\\ H=\frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33}-A_{13}^2)} \qquad \alpha=\frac{(A_{11}A_{33})^{1/2}-A_{13}}{A_{11}(\gamma_1+\gamma_2)} \qquad \beta=\frac{\gamma_3}{2\pi A_{44}}.\end{gathered} \tag{9}$$

Introduce the following in-plane stress components:

$$\sigma_1=\sigma_x+\sigma_y \qquad \sigma_2=\sigma_x-\sigma_y+2\mathrm{i}\tau_{xy} \qquad \tau_z=\tau_{zx}+\mathrm{i}\tau_{yz}. \tag{10}$$

This will simplify expressions (1), namely

$$\begin{gathered}\sigma_1=(A_{11}-A_{66})(\bar{\Lambda}u+\Lambda\bar{u})+2A_{13}\frac{\partial w}{\partial z} \qquad \sigma_2=2A_{66}\Lambda u\\ \sigma_2=\tfrac{1}{2}A_{13}(\bar{\Lambda}u+\Lambda\bar{u})+A_{33}\frac{\partial w}{\partial z} \qquad \tau_z=A_{44}\left(\frac{\partial u}{\partial z}+\Lambda w\right).\end{gathered} \tag{11}$$

We have now only four components of stress, instead of six, as it was in (1). The substitution of (6) in (11) yields

$$\begin{aligned}\sigma_1&=2A_{66}\frac{\partial^2}{\partial z^2}\{[\gamma_1^2-(1+m_1)\gamma_3^2]F_1+[\gamma_2^2-(1+m_2)\gamma_3^2]F_2\}\\ \sigma_2&=2A_{66}\Lambda^2(F_1+F_2+\mathrm{i}F_3)\\ \sigma_z&=A_{44}\frac{\partial^2}{\partial z^2}[(1+m_1)\gamma_1^2F_1+(1+m_2)\gamma_2^2F_2]\\ &=-A_{44}\Delta[(1+m_1)F_1+(1+m_2)F_2]\\ \tau_z&=A_{44}\Lambda\frac{\partial}{\partial z}[(1+m_1)F_1+(1+m_2)F_2+\mathrm{i}F_3].\end{aligned} \tag{12}$$

Here we used the fact that each F_k satisfies equation (7), and the relation $A_{11}\gamma_k^2-A_{13}m_k=A_{44}(1+m_k)$ (for $k=1,2$), which is an immediate consequence of (8).

Expressions (6) and (12) give a general solution, expressed in terms of three harmonic functions F_k. It is very attractive to express each function F_k through just *one* harmonic function as follows:

$$F_k(x, y, z) = c_k F(x, y, z) \tag{13}$$

where $z_k = z/\gamma_k$, and c_k is an as yet unknown complex constant. As we shall see further, this is possible indeed. All the results obtained in the paper are valid for isotropic solids, provided that we take

$$\begin{aligned} &\gamma_1 = \gamma_2 = \gamma_3 = 1 \qquad H = \frac{1-\nu^2}{\pi E} \qquad \alpha = \frac{1-2\nu}{2(1-\nu)} \\ &\beta = \frac{1+\nu}{\pi E} \qquad G_1 = \frac{(2-\nu)(1+\nu)}{\pi E} \qquad G_2 = \frac{\nu(1+\nu)}{\pi E} \end{aligned} \tag{14}$$

where E is the elastic modulus, and ν is the Poisson coefficient.

Consider a transversely isotropic elastic half-space $z \geqslant 0$. Let a point force, with components T_x, T_y, and P in Cartesian coordinates be applied at the point N_0 located at the boundary $z = 0$ of a transversely isotropic elastic half-space. We may assume, without loss of generality, that the polar cylindrical coordinates of N_0 are $(\rho_0, \phi_0, 0)$. We need to find the field of stresses and displacements at the point $M(\rho, \phi, z)$. Introduce the complex tangential force $T = T_x + \mathrm{i}T_y$. The general solution can be expressed through the three potential functions:

$$\begin{aligned} F_1 &= \frac{H\gamma_1}{m_1 - 1}[\tfrac{1}{2}\gamma_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1 + z_1)] \\ F_2 &= \frac{H\gamma_2}{m_2 - 1}[\tfrac{1}{2}\gamma_1(\bar{\Lambda}\chi_2 + \Lambda\bar{\chi}_2) + P\ln(R_2 + z_2)] \\ F_3 &= \mathrm{i}\frac{\gamma_3}{4\pi A_{44}}(\bar{\Lambda}\chi_3 - \Lambda\bar{\chi}_3). \end{aligned} \tag{15}$$

Here

$$\begin{aligned} &\chi_k(z) = \chi(z_k) \qquad R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z_k^2]^{1/2} \qquad \text{for } k = 1, 2, 3 \\ &\chi(z) = T[z\ln(R_0 + z) - R_0] \qquad R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2]^{1/2}. \end{aligned} \tag{16}$$

Substitution of (15) and (16) in (6) yields

$$\begin{aligned} u = {} & \frac{\gamma_3}{4\pi A_{44}}\left(\frac{T}{R_3} + \frac{q^2\bar{T}}{R_3(R_3 + z_3)^2}\right) + \frac{H\gamma_2}{m_2 - 1}\left[\frac{1}{2}\gamma_1\left(-\frac{T}{R_2} + \frac{q^2\bar{T}}{R_2(R_2 + z_2)^2}\right) + \frac{Pq}{R_2(R_2 + z_2)}\right] \\ & + \frac{H\gamma_1}{m_1 - 1}\left[\frac{1}{2}\gamma_2\left(-\frac{T}{R_1} + \frac{q^2\bar{T}}{R_1(R_1 + z_1)^2}\right) + \frac{Pq}{R_1(R_1 + z_1)}\right] \end{aligned} \tag{17}$$

$$\begin{aligned} w = H & \left[\tfrac{1}{2}(T\bar{q} + \bar{T}q)\left(\frac{\gamma_2 m_1}{(m_1 - 1)R_1(R_1 + z_1)} + \frac{\gamma_1 m_2}{(m_2 - 1)R_2(R_2 + z_2)}\right)\right. \\ & \left. + P\left(\frac{m_1}{(m_1 - 1)R_1} + \frac{m_2}{(m_2 - 1)R_2}\right)\right]. \end{aligned} \tag{18}$$

Here

$$q = \rho\, e^{i\phi} - \rho_0\, e^{i\phi_0}. \tag{19}$$

We consider a transversely isotropic elastic half-space $z \geqslant 0$ (figure 1). A flat circular punch of radius a is bonded to its boundary $z = 0$, with the punch centre coinciding with the coordinate system origin $\rho = 0$. Let a point force N be applied in the Oz direction at the point with the polar cylindrical coordinates (ρ, ϕ, z). We may assume, without loss of generality, that $\phi = 0$. We need to find the punch settlement w_N, its tangential displacement u_N, and the angle of inclination δ_N which are due to the point load N. The reader is reminded that the punch settlement is understood to be the normal displacement of the punch centre; the angle of inclination is the angle between the punch base and the plane $z = 0$.

First of all, we need to solve two auxiliary problems: one is the centrally loaded bonded punch, and the second one is the problem of an inclined bonded punch. We consider below each problem separately, after which the reciprocal theorem is used to obtain the solution to the main problem.

Problem 1. We consider the mixed-mixed problem characterised by the following boundary conditions:

$$\begin{array}{llll} u = 0 & \text{for} & 0 \leqslant \rho \leqslant a & 0 \leqslant \phi < 2\pi \\ w = w_0 & \text{for} & 0 \leqslant \rho \leqslant a & 0 \leqslant \phi < 2\pi \\ \sigma = 0 & \text{for} & a \leqslant \rho \leqslant \infty & 0 \leqslant \phi < 2\pi \\ \tau = 0 & \text{for} & a \leqslant \rho \leqslant \infty & 0 \leqslant \phi < 2\pi \end{array} \tag{20}$$

The solution to the problem may be presented in the form (Fabrikant 1986)

$$\sigma(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\int_\rho^a \frac{f_1(t)\,t\,dt}{(t^2-\rho^2)^{1/2}} \qquad \tau_\rho(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{d}{d\rho}\int_\rho^a \frac{f_2(t)\,dt}{(t^2-\rho^2)^{1/2}}. \tag{21}$$

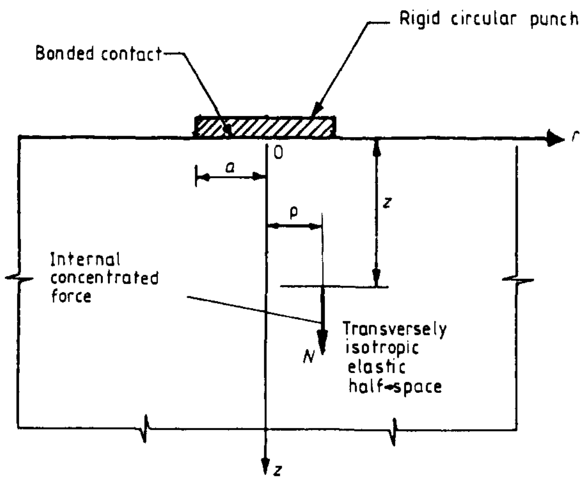

Figure 1.

Here σ is the normal traction exerted by the punch, $\tau_\rho = \tau_{\rho z} + i\tau_{\theta z}$, and the stress functions f_1 and f_2 are defined in this particular case by

$$
\begin{aligned}
f_1(x) &= -\frac{w_0}{\pi^2 H}\cosh \pi\theta \cos\left(\theta \ln\frac{a+x}{a-x}\right)\\
f_2(x) &= -\frac{w_0}{\pi^2 H}\cosh \pi\theta \sin\left(\theta \ln\frac{a+x}{a-x}\right)\\
\theta &= \frac{1}{2\pi}\ln\frac{\sqrt{\gamma_1\gamma_2}+\alpha}{\sqrt{\gamma_1\gamma_2}-\alpha}
\end{aligned}
\tag{22}
$$

where w_0 is the punch settlement. In the case of isotropy, the value of θ is defined by $\theta = (1/2\pi)\ln(3-4\nu)$.

Now we need to substitute formulae (21) in (15), modified for the case of distributed loading, and to compute the integrals involved. Here are some detailed of the derivation. Substitution of the first expression (21) in (15) leads to the integral

$$
I_1 = \int_0^{2\pi}\int_0^a \ln(R_0+z)\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 \frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^a \frac{f_1(x)x\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}. \tag{23}
$$

By interchanging the order of integration in (23), we obtain

$$
I_1 = -\int_0^a f_1(x)\,\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^x \frac{\ln(R_0+z)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{24}
$$

The double integral in (24) can be computed by using (A1)-(A4), with the result

$$
I_1 = -2\pi\int_0^a f_1(x)\ln\{l_2(x)+[l_2^2(x)-\rho^2]^{1/2}\}\,\mathrm{d}x.
$$

The following notation is used throughout the paper:

$$
\begin{aligned}
l_1(x,\rho,z) &\equiv l_1(x) = \tfrac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}-[(\rho-x)^2+z^2]^{1/2}\}\\
l_2(x,\rho,z) &\equiv l_2(x) = \tfrac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}+[(\rho-x)^2+z^2]^{1/2}\}.
\end{aligned}
\tag{25}
$$

The abbreviations l_1 and l_2 everywhere stand for $l_1(a)$ and $l_2(a)$ respectively. The notation $l_{1k}(x)$ and $l_{2k}(x)$ is understood as $l_1(x,\rho,z_k)$ and $l_2(x,\rho,z_k)$ respectively, for $k=1,2$.

When substituting the second expression of (21) in (15), we have to remember the relationship between $\tau = \tau_{zx}+i\tau_{yz}$ and $\tau_\rho = \tau_{\rho z}+i\tau_{\theta z}$, namely, $\tau = \tau_\rho\,e^{i\phi}$. The substitution leads to the integral

$$
I_2 = \Lambda\int_0^{2\pi}\int_0^a [z\ln(R_0+z)-R_0]\,e^{-i\phi_0}\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^a\frac{f_2(x)\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}.
$$

Again, interchanging the order of integration, we obtain

$$
I_2 = -\Lambda\int_0^a f_2(x)\frac{\mathrm{d}x}{x}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^x [z\ln(R_0+z)-R_0]\,e^{-i\phi_0}\frac{\rho_0^2\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{26}
$$

The double integral in (26) can be computed according to (A7)-(A10), and the final result is rather simple

$$I_2 = -2\pi \int_0^a f_2(x) \sin^{-1} \frac{x}{l_2(x)} \, dx. \tag{27}$$

Now the potential functions (15) can be expressed through the stress functions as follows:

$$\begin{aligned}
F_1 &= -\frac{2\pi H}{m_1 - 1} \left\{ \gamma_1 \int_0^a f_1(x) \ln[l_{21}(x) + (l_{21}^2(x) - \rho^2)^{1/2}] \, dx \right. \\
&\quad \left. + \sqrt{\gamma_1 \gamma_2} \int_0^a f_2(x) \sin^{-1} \left(\frac{x}{l_{21}(x)} \right) dx \right\} \\
F_2 &= -\frac{2\pi H}{m_2 - 1} \left\{ \gamma_2 \int_0^a f_1(x) \ln[l_{22}(x) + (l_{22}^2(x) - \rho^2)^{1/2}] \, dx \right. \\
&\quad \left. + \sqrt{\gamma_1 \gamma_2} \int_0^a f_2(x) \sin^{-1} \left(\frac{x}{l_{22}(x)} \right) dx \right\} \\
F_3 &= 0.
\end{aligned} \tag{28}$$

Taking into consideration (22), formulae (28) may be rewritten as

$$\begin{aligned}
F_1 &= \frac{2w_0 \cosh(\pi\theta)}{\pi(m_1 - 1)} \left\{ \gamma_1 \int_0^a Y_c(x) \ln[l_{21}(x) + (l_{21}^2(x) - \rho^2)^{1/2}] \, dx \right. \\
&\quad \left. + \sqrt{\gamma_1 \gamma_2} \int_0^a Y_s(x) \sin^{-1} \left(\frac{x}{l_{21}(x)} \right) dx \right\} \\
F_2 &= \frac{2w_0 \cosh(\pi\theta)}{\pi(m_2 - 1)} \left\{ \gamma_2 \int_0^a Y_c(x) \ln[l_{22}(x) + (l_{22}^2(x) - \rho^2)^{1/2}] \, dx \right. \\
&\quad \left. + \sqrt{\gamma_1 \gamma_2} \int_0^a Y_s(x) \sin^{-1} \left(\frac{x}{l_{22}(x)} \right) dx \right\} \\
F_3 &= 0
\end{aligned} \tag{29}$$

where w_0 is the punch settlement, θ is defined by (22) and

$$Y_c(x) = \cos\left(\theta \ln \frac{a + x}{a - x} \right) \qquad Y_s(x) = \sin\left(\theta \ln \frac{a + x}{a - x} \right). \tag{30}$$

Substitution of (29) in (6) and (11) gives a complete solution to the first auxiliary problem. We need only the expression for the normal displacement

$$\begin{aligned}
w(\rho, z) &= \frac{2}{\pi} w_0 \cosh(\pi\theta) \sum_{k=1}^{2} \frac{m_k}{m_k - 1} \\
&\quad \times \left\{ \int_0^a \frac{[l_{2k}^2(x) - x^2]^{1/2}}{l_{2k}^2(x) - l_{2k}^2(x)} Y_c(x) \, dx \right. \\
&\quad \left. - \frac{\sqrt{\gamma_1 \gamma_2}}{\gamma_k} \int_0^a \frac{[x^2 - l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x) - l_{1k}^2(x)} Y_s(x) \, dx \right\}.
\end{aligned} \tag{31}$$

Taking into consideration the relationship between the punch settlement w_0 and the applied to the punch force P (Fabrikant 1989)

$$w_0 = \frac{PH \tanh(\pi\theta)}{2a\theta} \tag{32}$$

expression (31) can be rewritten as

$$\begin{aligned} w(\rho, z) = {} & \frac{PH \sinh(\pi\theta)}{\pi a\theta} \sum_{k=1}^{2} \frac{m_k}{m_k - 1} \\ & \times \left\{ \int_0^a \frac{[l_{2k}^2(x) - x^2]^{1/2}}{l_{2k}^2(x) - l_{1k}^2(x)} Y_c(x)\, dx \right. \\ & \left. - \frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k} \int_0^a \frac{[x^2 - l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x) - l_{1k}^2(x)} Y_s(x)\, dx \right\}. \end{aligned} \tag{33}$$

Problem 2. The boundary conditions in the case of a flat circular bonded punch subjected to a shifting force T and a tilting moment M are

$$\begin{array}{llll} u = u_0 = \text{constant} & \text{for} & 0 \leqslant \rho \leqslant a & 0 \leqslant \phi < 2\pi \\ w = -\delta\rho \cos\phi & \text{for} & 0 \leqslant \rho \leqslant a & 0 \leqslant \phi < 2\pi \\ \sigma = 0 & \text{for} & a \leqslant \rho \leqslant \infty & 0 \leqslant \phi < 2\pi \\ \tau = 0 & \text{for} & a \leqslant \rho \leqslant \infty & 0 \leqslant \phi < 2\pi. \end{array} \tag{34}$$

Here u_0 is the tangential displacement of the punch, and δ is the angle of inclination. Again, by using a method similar to that used in problem 1, we may verify that the equations (7) and the boundary conditions (34) can be satisfied by the potential functions

$$\begin{aligned} F_1 = {} & \frac{4\pi H}{m_1 - 1} \frac{\cos\phi}{\rho} \left\{ \alpha \int_0^a \{z_1 - [l_{21}^2(x) - x^2]^{1/2}\} f(x)\, dx \right. \\ & \left. + \gamma_1 \int_0^a \{x - [x^2 - l_{11}^2(x)]^{1/2}\} f_1(x) \right\}, \\ F_2 = {} & \frac{4\pi H}{m_2 - 1} \frac{\cos\phi}{\rho} \left\{ \alpha \int_0^a \{z_2 - [l_{22}^2(x) - x^2]^{1/2}\} f(x)\, dx \right. \\ & \left. + \gamma_2 \int_0^a \{x - [x^2 - l_{12}^2(x)]^{1/2}\} f_1(x) \right\}, \end{aligned} \tag{35}$$

$$\begin{aligned} F_3 = {} & D \frac{2\gamma_3}{A_{44}} \frac{\sin\phi}{\rho} \int_0^a \{z_3 - [l_{23}^2(x) - x^2]^{1/2}\}\, dx \\ = {} & D \frac{2\gamma_3}{A_{44}} \frac{\sin\phi}{\rho} \left[z_3 a - \frac{(l_{23}^2 - a^2)^{1/2}(2a^2 - l_{13}^2)}{2a} - \frac{\rho^2}{2} \sin^{-1}\left(\frac{a}{l_{23}}\right) \right] \end{aligned}$$

Here f and f_1 are the stress functions and D is a constant. They are defined (Fabrikant 1989) as follows:

$$f(t) = -\frac{\delta\cosh^2(\pi\theta)}{\pi^2 H\sinh(\pi\theta)}\left[tY_s(t) - \theta a Y_c(t)\right] + AY_c(t) \tag{36}$$

$$f_1(t) = \frac{\delta\cosh(\pi\theta)}{\pi^2 H}\left[tY_c(t) + \theta a Y_s(t)\right] + \tanh(\pi\theta)AY_s(t) \tag{37}$$

$$D = \frac{\pi\theta\alpha}{\gamma_1\gamma_2\sinh(\pi\theta)}A \qquad \tanh(\pi\theta) = \frac{\alpha}{\sqrt{\gamma_1\gamma_2}} \tag{38}$$

$$A = \left(u_0 + \frac{\delta a\theta\alpha}{\tanh(\pi\theta)}\right)\left[\frac{\pi^2 H\alpha}{\cosh(\pi\theta)}\left(1 + \frac{\pi\theta(G_1+G_2)}{\tanh(\pi\theta)(G_1-G_2)}\right)\right]^{-1}. \tag{39}$$

Substitution of (35) in (6) and (11) gives a complete solution to the second auxiliary problem. We need only the expression for the normal displacement

$$\begin{aligned} w(\rho, z) = {} & 4\pi H\frac{\cos\phi}{\rho} \\ & \times\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\alpha}{\gamma_k}\int_0^a\left(1 - \frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\right)f(x)\,\mathrm{d}x\right.\right. \\ & \left.\left. - \int_0^a\frac{l_{1k}(x)[\rho^2-l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,\mathrm{d}x\right]\right\}. \end{aligned} \tag{40}$$

The displacements of the punch are related to the applied loading as (Fabrikant 1989)

$$\begin{aligned} u_0 &= \frac{1}{8a}\left[\pi(G_1+G_2) + \frac{(1+4\theta^2)\tanh(\pi\theta)}{\theta(1+\theta^2)}(G_1-G_2)\right]T - \frac{3H\alpha}{4a^2(1+\theta^2)}M \\ \delta &= \frac{3H\alpha}{4a^2(1+\theta^2)}\left(-T + \frac{M}{a\theta\sqrt{\gamma_1\gamma_2}}\right). \end{aligned} \tag{41}$$

The main problem. Now we may apply the reciprocal theorem in order to obtain the punch displacements due to a normal concentrated force N applied at the point $(\rho, 0, z)$. The normal displacement of the punch is readily available from (33) as

$$\begin{aligned} w_N = {} & \frac{NH\sinh(\pi\theta)}{\pi a\theta}\sum_{k=1}^{2}\frac{m_k}{m_k-1} \\ & \times\left\{\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}Y_c(x)\,\mathrm{d}x\right. \\ & \left. - \frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}Y_s(x)\,\mathrm{d}x\right\}. \end{aligned} \tag{42}$$

In order to find the tangential displacement of the punch, we have to apply a unit tangential force T in the positive Ox direction. From (36)-(39) and (41), one can find the stress functions as

$$\begin{aligned} f(x) &= \frac{T\sqrt{\gamma_1\gamma_2}\cosh(\pi\theta)}{4\pi^2 a(1+\theta^2)}\left(3\frac{x}{a}Y_s(x) + \frac{1-2\theta^2}{\theta}Y_c(x)\right) \\ f_1(x) &= \frac{T\sqrt{\gamma_1\gamma_2}\sinh(\pi\theta)}{4\pi^2 a(1+\theta^2)}\left(-3\frac{x}{a}Y_c(x) + \frac{1-2\theta^2}{\theta}Y_s(x)\right) \end{aligned} \tag{43}$$

and the tangential displacements can be defined from (40) in the form

$$u_N = \frac{NH\sqrt{\gamma_1\gamma_2}\sinh(\pi\theta)}{\pi a(1+\theta^2)\rho} \sum_{k=1}^{2} \left\{ \frac{m_k}{m_k-1} \left[\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k} \int_0^a \left(1 - \frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\right) \right.\right.$$
$$\times \left(3\frac{x}{a} Y_s(x) + \frac{1-2\theta^2}{\theta} Y_c(x)\right) \mathrm{d}x - \int_0^a \frac{l_{1k}(x)[\rho^2 - l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)} \tag{44}$$
$$\left.\left. \times \left(-3\frac{x}{a} Y_c(x) + \frac{1-2\theta^2}{\theta} Y_s(x)\right) \mathrm{d}x \right]\right\}.$$

We need to apply to the punch a unit tilting moment M in order to find the angular displacement δ. The stress functions in this case are

$$f(x) = -\frac{3M\cosh(\pi\theta)}{4\pi^2 a^3\theta(1+\theta^2)}(xY_s(x) - \theta a Y_c(x))$$
$$f_1(x) = \frac{3M\sinh(\pi\theta)}{4\pi^2 a^3\theta(1+\theta^2)}(xY_c(x) + \theta a Y_s(x)) \tag{45}$$

and the angular displacement will take the form

$$\delta_N = \frac{3NH\sinh(\pi\theta)}{\pi a^3\theta(1+\theta^2)\rho}$$
$$\times \sum_{k=1}^{2} \left\{ \frac{m_k}{m_k-1} \left[\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k} \int_0^a \frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)} (xY_s(x) - \theta a Y_c(x))\,\mathrm{d}x \right.\right. \tag{46}$$
$$\left.\left. - \int_0^a \frac{l_{1k}(x)[\rho^2 - l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)} (xY_c(x) + \theta a Y_s(x))\,\mathrm{d}x \right]\right\}.$$

Formulae (42), (44) and (46) are the main new results of this paper.

3. Numerical example

It is of interest to compare the influence of a concentrated load on a bonded punch to that of a smooth punch. The reader is reminded that the term smooth punch is used to indicate the type of punch which does not exert any tangential stress in the domain of contact. These two cases represent two extremes in the interaction between a punch and an elastic half-space, so the normal and angular displacements of a smooth punch due to a point load N applied at the point $(\rho, 0, z)$ are (Fabrikant 1989)

$$w_N = \frac{NH}{a} \sum_{k=1}^{2} \frac{m_k}{m_k-1} \sin^{-1}\left(\frac{a}{l_{2k}}\right) \tag{47}$$

$$\delta_N = -\frac{3NH}{2a^3} \sum_{k=1}^{2} \left\{ \frac{m_k}{m_k-1} \left[\rho\sin^{-1}\left(\frac{a}{l_{2k}}\right) - \frac{l_{1k}(l_{2k}^2 - a^2)^{1/2}}{l_{2k}} \right]\right\}. \tag{48}$$

One should note that the solutions (42), (46) and (47), (48) coincide in the case of $\theta = 0$. In the case of isotropy, this corresponds to the Poisson coefficient $\nu = \frac{1}{2}$. The greatest difference between the solutions for a bonded and a smooth punch is attained for the Poisson coefficient $\nu=0$. This value was used in numerical computations. Formulae (47) and (48) in the case of isotropy will take the form

$$w_N=\frac{NH}{a}\left(\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{z(a^2-l_1^2)^{1/2}}{2(1-\nu)(l_2^2-l_1^2)}\right) \tag{49}$$

$$\delta_N=\frac{3NH}{2a^3}\rho\left(\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{a(l_2^2-a^2)^{1/2}}{l_2^2}+\frac{za^2(a^2-l_1^2)^{1/2}}{(1-\nu)l_2^2(l_2^2-l_1^2)}\right). \tag{50}$$

The limiting cases of (42) and (46) in the case of isotropy are

$$\begin{aligned}w_N=\frac{NH\sinh(\pi\theta)}{\pi a\theta}\Bigg[&\int_0^a\left(\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\right.\\&\left.-\frac{z^2[x^2(2x^2+2z^2-\rho^2)-l_2^4(x)]}{2(1-\nu)[l_2^2(x)-x^2]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right)Y_c(x)\,dx\\&-\frac{1}{2(1-\nu)}\int_0^a\left((1-2\nu)\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}\right.\\&\left.+\frac{z^2[x^2(2x^2+2z^2)-\rho^2)-l_1^4(x)]}{[x^2-l_1^2(x)]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right)Y_s(x)\,dx\Bigg]\end{aligned} \tag{51}$$

$$\begin{aligned}\delta_N=-\frac{3NH\sinh(\pi\theta)}{\pi a^3\theta(1+\theta^2)\rho}\Bigg[&\frac{1}{2(1-\nu)}\int_0^a\left((1-2\nu)\frac{l_2(x)[l_2^2(x)-\rho^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\right.\\&\left.+\frac{z\rho^2[l_2^2(x)-x^2]^{1/2}}{[l_2^2(x)-l_1^2(x)]^3}[4x^2-3l_1^2(x)-l_2^2(x)]\right)[a\theta Y_c(x)-xY_s(x)]\,dx\\&+\int_0^a\left(\frac{l_1(x)[\rho^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}-\frac{z\rho^2[x^2-l_1^2(x)]^{1/2}}{2(1-\nu)[l_2^2(x)-l_1^2(x)]^3}\right.\\&\left.\times[4x^2-l_1^2(x)-3l_2^2(x)]\right)[xY_c(x)+a\theta Y_s(x)]\,dx\Bigg].\end{aligned} \tag{52}$$

The results of computations are presented in figures 2 and 3. The full curves give the results for a bonded punch, while the broken curves give the corresponding data for a smooth punch. Figure 2 plots the dimensionless parameter w_N/w^0 against ρ/a computed due to (49) and (51) for the values of $z=0, 0.1, 0.5$ and 1. The parameter $w^0=\pi NH/(2a)$ corresponds to the settlement of a smooth punch subjected to a central loading equal to N. The plot shows that the settlement of a smooth punch is always greater (up to about $0.1w^0$) than that of a bonded punch.

The plot of the magnitude of δ_N/δ^0 against ρ/a is given in figure 3. Computations were made due to (50) and (52) for $z=0, 0.1, 0.5$ and 1, with $\delta^0=3\pi NH/(4a^2)$ giving the maximum angle of inclination of a smooth punch. The difference between the results for a smooth punch and a bonded punch does not exceed $0.08\delta^0$. All the computations were made for the Poisson coefficient $\nu=0$. Since in real materials $\nu>0$, the difference between the smooth and bonded punch solutions will be smaller than that indicated above.

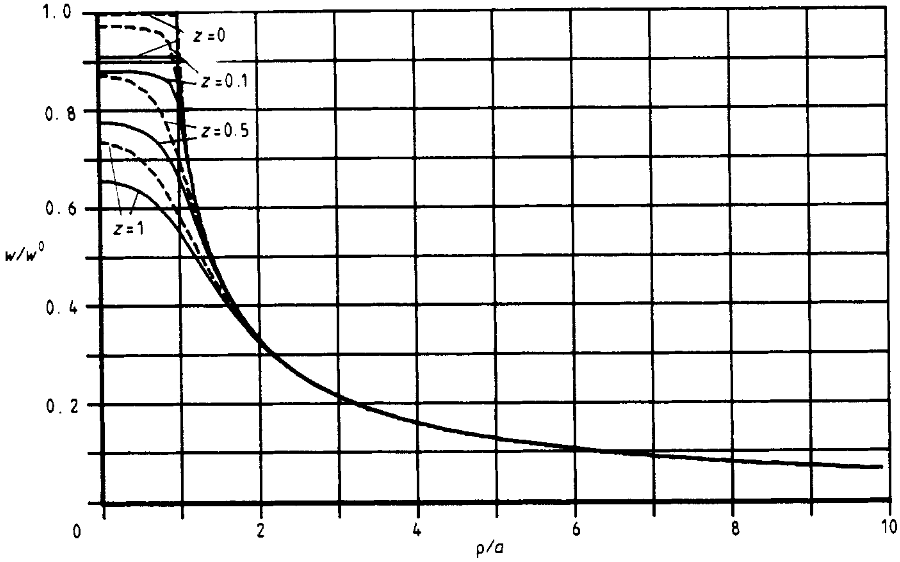

Figure 2.

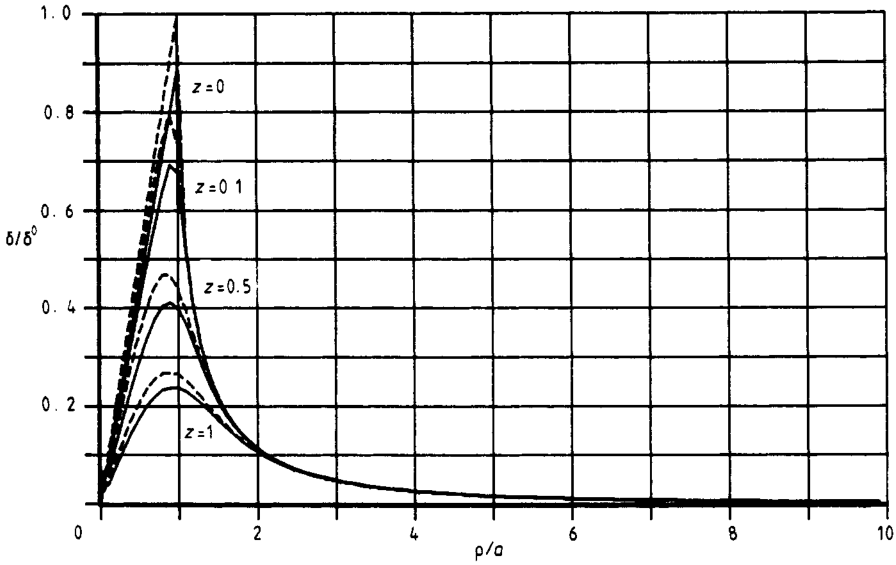

Figure 3.

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Appendix

The following integrals may be computed by the method described by Fabrikant (1989)

$$I=\int_0^{2\pi}\int_0^a \frac{z\ln(R_0+z)-R_0}{(a^2-\rho_0^2)^{1/2}}\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0=\pi\left[\left(z^2-a^2-\frac{\rho^2}{2}\right)\sin^{-1}\left(\frac{a}{l_2}\right)\right.$$
$$\left.-\frac{3(2a^2-l_1^2)}{2a}(l_2^2-a^2)^{1/2}+2az\ln[l_2+(l_2^2-\rho^2)^{1/2}\right] \tag{A1}$$

$$\frac{\partial I}{\partial z}=2\pi\left(z\sin^{-1}\left(\frac{a}{l_2}\right)-(a^2-l_1^2)^{1/2}+a\ln[l_2+(l_2^2-\rho^2)^{1/2}]\right) \tag{A2}$$

$$\frac{\partial^2 I}{\partial z^2}=2\pi\sin^{-1}\left(\frac{a}{l_2}\right) \tag{A3}$$

$$\frac{\partial^2 I}{\partial z\partial a}=2\pi\ln[l_2+(l_2^2-\rho^2)^{1/2}] \tag{A4}$$

$$\frac{\partial I}{\partial a}=2\pi\left(z\ln[l_2+(l_2^2-\rho^2)^{1/2}]-a\sin^{-1}\left(\frac{a}{l_2}\right)-(l_2^2-a^2)^{1/2}\right). \tag{A5}$$

Yet another important integral is

$$J=\int_0^{2\pi}\int_0^a [z\ln(R_0+z)-R_0]\frac{\mathrm{e}^{-\mathrm{i}\phi_0}\rho_0^2\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{(a^2-\rho_0^2)^{1/2}}$$
$$=\pi\rho\,\mathrm{e}^{-\mathrm{i}\phi}\left\{\left(\frac{a^2}{2}+\frac{z^2}{2}-\frac{\rho^2}{8}\right)\sin^{-1}\left(\frac{a}{l^2}\right)-\frac{2za^3}{3\rho^2}\right.$$
$$\left.+\left(\frac{5l_1^2}{8a}-\frac{a}{2}+\frac{2a^3}{3\rho^2}-\frac{al_1^2}{3\rho^2}-\frac{l_1^4}{12a\rho^2}\right)(l_2^2-a^2)^{1/2}\right\}. \tag{A6}$$

Several partial derivatives may be computed as follows:

$$\frac{\partial J}{\partial z}=\int_0^{2\pi}\int_0^a \ln(R_0+z)\frac{\mathrm{e}^{-\mathrm{i}\phi_0}\rho_0^2\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{(a^2-\rho_0^2)^{1/2}}$$
$$=\pi\rho\,\mathrm{e}^{-\mathrm{i}\phi}\left[z\sin^{-1}\left(\frac{a}{l_2}\right)-(a^2-l_1^2)^{1/2}\left(1-\frac{l_1^2+2a^2}{3\rho^2}\right)-\frac{2a^3}{3\rho^2}\right] \tag{A7}$$

$$\frac{\partial^2 J}{\partial z^2}=\int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{-\mathrm{i}\phi_0}\rho_0^2\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{R_0(a^2-\rho_0^2)^{1/2}}=\pi\rho\,\mathrm{e}^{-\mathrm{i}\phi}\left[\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{a}{l_2^2}(l_2^2-a^2)^{1/2}\right] \tag{A8}$$

$$\Lambda J=-\int_0^{2\pi}\int_0^a \frac{q\,\mathrm{e}^{-\mathrm{i}\phi_0}\rho_0^2\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{(R_0+z)(a^2-\rho_0^2)^{1/2}}$$
$$=\pi\left[\left(a^2+z^2-\frac{\rho^2}{2}\right)\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{2a^2-3l_1^2}{2a}(l_2^2-a^2)^{1/2}\right] \tag{A9}$$

It is recalled that q is defined by (19):

$$\frac{\partial}{\partial a}\Lambda J=-\frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{q\,\mathrm{e}^{-\mathrm{i}\phi_0}\rho_0^2\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{(R_0+z)(a^2-\rho_0^2)^{1/2}}=2\pi a\sin^{-1}\left(\frac{a}{l_2}\right). \tag{A10}$$

Here are some derivatives used in the paper:

$$\frac{\partial}{\partial a}\sin^{-1}\left(\frac{a}{l_2}\right)=\frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2} \qquad \frac{\partial}{\partial a}\ln[l_2+(l_2^2+\rho^2)^{1/2}]=\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2} \tag{A11}$$

$$\frac{\partial}{\partial z}\sin^{-1}\left(\frac{a}{l_2}\right)=-\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2} \qquad \frac{\partial}{\partial z}\ln[l_2+(l_2^2-\rho^2)^{1/2}]=\frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2} \tag{A12}$$

$$\Lambda\sin^{-1}\left(\frac{a}{l_2}\right)=-\frac{l_1(l_2^2-a^2)^{1/2}}{l_2[l_2^2-l_1^2]}\mathrm{e}^{\mathrm{i}\phi} \tag{A13}$$

$$\Lambda^2\sin^{-1}\left(\frac{a}{l_2}\right)=\frac{a\,\mathrm{e}^{2\mathrm{i}\phi}(l_2^2-a^2)^{1/2}}{l_2^2[l_2^2-l_1^2]^3}[3\rho^2l_2^2+\rho^2l_1^2-6a^2\rho^2+2l_1^4] \tag{A14}$$

$$\Lambda\ln[l_2+(l_2^2-\rho^2)^{1/2}]=\frac{\mathrm{e}^{\mathrm{i}\phi}}{\rho}\left[1-\frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2}\right] \tag{A15}$$

$$\begin{aligned}&\Lambda^2\ln[l_2+(l_2^2-\rho^2)^{1/2}]\\&\quad=-\frac{2\,\mathrm{e}^{2\mathrm{i}\phi}}{\rho^2}-\frac{a\,\mathrm{e}^{2\mathrm{i}\phi}(a^2-l_1^2)^{1/2}}{l_1^2(l_2^2-l_1^2)^3}[6a^2\rho^2-2l_2^4-\rho^2l_2^2-3\rho^2l_1^2]\end{aligned} \tag{A16}$$

$$\frac{\partial^2}{\partial z^2}\sin^{-1}\left(\frac{a}{l_2}\right)=-\frac{\partial}{\partial z}\left(\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\right)=\frac{z[a^2(2a^2+2z^2-\rho^2)-l_1^4]}{(a^2-l_1^2)^{1/2}(l_2^2-l_1^2)^3} \tag{A17}$$

$$\frac{\partial^2}{\partial z^2}\ln[l_2+(l_2^2-\rho^2)^{1/2}]=\frac{\partial}{\partial z}\left(\frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\right)=\frac{z[a^2(2a^2+2z^2-\rho^2)-l_2^4]}{(l_2^2-a^2)^{1/2}(l_2^2-l_1^2)^3} \tag{A18}$$

$$\Lambda\frac{\partial}{\partial z}\sin^{-1}\left(\frac{a}{l_2}\right)=-\Lambda\left(\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\right)=\frac{\rho\,\mathrm{e}^{\mathrm{i}\phi}(a^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)^3}[3l_2^2-l_1^2-4a^2] \tag{A19}$$

$$\Lambda\frac{\partial}{\partial z}\ln[l_2+(l_2^2-\rho^2)^{1/2}]=\Lambda\left(\frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\right)=\frac{\rho\,\mathrm{e}^{\mathrm{i}\phi}(l_2^2-a^2)^{1/2}}{(l_2^2-l_1^2)^3}[4a^2-l_2^2-3l_1^2]. \tag{A20}$$

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